4

What Do Math Teachers Need to Be?

HERBERT CLEMENS

Here's a problem:

$$\frac{21}{1320} + \frac{12}{1255} = ?$$

Give your answer as a fraction in lowest terms.

What is your reaction? My guess is that the reaction of many grade school teachers would be something like, "Those numbers are too big, and, anyway, what book did that problem come from?"

Here's another problem:

$$\frac{21^{-2}}{7^{-3}} \div \frac{\frac{1}{7} - \frac{4}{21}}{1 + \frac{5}{7}} = ?$$

Give your answer as a fraction in lowest terms.

What is your reaction? Perhaps this time the first comment of elementary school teachers might be, "We're not supposed to know this, are we?" As a mathematician with absolutely no formal training in education, I can only guess at what a teacher's reaction to these problems would be, but I can tell you for sure what my first reaction is, namely, "I can do these problems!" Would I like to do 50 of these kinds of problems? Probably not. (Maybe I wouldn't even like to do five, but one or two would be okay.) When pressed by the chemistry of a school classroom to reach out to kids, my next reaction might be (and often has been)

> We can probably fool around with this big messy problem and pull together some interesting things about math. In fact, I can probably use this problem to explain some things better than the book can . . . let's see how would I start . . . it doesn't

matter if that approach doesn't work for some of the kids . . . after I read their minds a bit, I'll reorganize the problem with them in a way that works for them.

I like teaching and I like the way I teach; otherwise I wouldn't have written this chapter. So, to begin my answer to the question posed in the title, the first thing I think a math teacher must be is what I like to think I am (on my good days), and that is *unafraid*. To explain a bit more, let's ask the reverse question, What does it mean to be afraid? It means being anxious when confronted with that which is unknown or unfamiliar; it means jumping to the assumption, "I cannot deal with this new person or thing, and so this encounter is going to be a negative, defeating personal experience." If that's what we mean by being afraid, then many school teachers I know are afraid of math.

What else awould a teacher be? Again, let's pose a question: Are the above problems "fun"? Quite frankly, no. Can learning to solve them be fun? Maybe in one specially configured circumstance, yes, but not over the long haul. The numbers are big, the necessary multiplications and additions get very boring very quickly (unless we use a calculator), and, all in all, there are a lot more fun ways to spend an afternoon. I'm much too pseudo-sophisticated and cynical to be won over by the "math-is-fun" crowd. Interesting? Quite often, yes. But fun? No. Doing math is a lot like developing a spiritual component to life—when you are young and immature you do it because your parents force you to; when you grow up you understand its importance in your own life and you do it in your own personal way. The enterprise of mathematics is too basic and important, with too much beauty, history, and depth, to be trivialized into "fun and games."

So the second thing a math teacher must be is what any good cleric must be, namely, *reverent*. If we are good math teachers, then by our demeanor and way of talking about it, we reflect respect and reverence—not some ostentatious false piety, but a belief that the enterprise is worth it. True respect and reverence for mathematics, as for religion, rests on the belief that its content is solid enough to be doubted, questioned, probed, and attacked. The outcome is usually that the basic truths are unmoved, but that the individual who has the integrity and intelligence to question, probe, and attack has learned and grown in the process. We cannot expect young, immature minds to be capable of appreciating much of the beauty and depth, but we really must configure experiences—even when we have to force them a bit—that open up for our children the opportunity to appreciate these things in adult life. The question, "We're not supposed to know this, are we?" is the mathematical equivalent to some of the questions that were current during my adawescent religious training, like "What kind of a sin is it to French kiss for one minute and 30 seconds?"

What comes next in this scout's oath of math teacher virtues? I'd say it's the realization that, in the area of mathematical culture, the United States is an underdeveloped nation. All measures of the ability of our school children and general populace to deal with mathematics rank us far down on the roster of nations. Our power in advanced research and technology is derived increasingly from scholars who receive their basic education in other mathematically developed countries. So our math teachers, and all the rest of us, should be *humble*. We're a poor mathematical nation, we're ignorant, and the most important first step in attacking ignorance is to admit to ourselves that we are ignorant. In the fight for mathematical literacy, we're the underdogs. This may give us energy and determination, which is all to the good, but it should also induce a certain quiet sense of shame, and of admiration of those who do better than we do.

What's next? Let's go back to the reaction "I can do this problem."

$$\frac{21^{-2}}{7^{-3}} \div \frac{\frac{1}{7} - \frac{4}{21}}{1 + \frac{5}{7}} = ?$$

In fact, this problem is a terrific opportunity! It is complicated, like most interesting things, and it demands that we analyze it, break it up into simpler "steps." For example, I can work with

$$\frac{\frac{1}{7} - \frac{4}{21}}{1 + \frac{5}{7}}.$$

I know that I don't change the big ratio if I multiply it by

$$1 = \frac{21}{21}.$$

This problem is an opportunity to talk about clever ways to write the number 1. Many, many complicated fraction problems get a lot simpler if one writes "one" in a helpful way, but learning how to do that comes from "getting the hang of it," not from learning a bunch of rules. So a math teacher must be *opportunistic* and must pick the right moment to do the right thing.

In order to exploit opportunities, a math teacher must be *versatile*. Children are so different, one from another, in the ways they think, visualize, and learn. A teacher doesn't need one technique to teach fractions or place value; he or she needs three or four. He or she needs a file cabinet full of different materials, needs to be able to keep two or three different approaches going at the same time in the same lesson, and needs to have alternative approaches available if the current one isn't working. In my experience, no particular technique for presenting a given lesson on a given day reaches more than four or five students out of a group of 24. (Fortunately it's not always the same group of four or five!)

All these virtues are part of feeling in control of one's math. All good math teachers are *in control* of their math—ultimately this is reflected in the ability to change the rules if they need to and to be able to satisfy themselves and others as to whether the change is legitimate. To give a couple of very concrete examples of what I mean by being "in control," we might write, "there is a number whose square is -2," or we might write, "10101 = 21." Both of these are perfectly good mathematical statements if we explain them appropriately.[1] So we drive the mathematics, not the reverse. We're better than the book, or at least we can be, if we put our minds to it!

So far I've said nothing about what mathematics a math teacher should know, and that's not because I think that the question is unimportant. I believe I have mentioned those qualities, or lack thereof, that distinguish teachers, and the population in general, in mathematically developed countries from those in our own country. Empirical evidence indicates that these factors may be relevant to math education.

But what about subject matter? It is very appealing to think that there is a set of quantitative skills that we can give to math teachers and, together with some good teaching technique, they will do the job. There are some problems with that. For example, the little I know about research in math education says that better results come from teachers who inspire and who command respect for their quantitative ability than from teachers who know some particular type of mathematics, however enlightened and well designed.

This is not to say that the teacher's command of the subject is unimportant—it is essential. I can't imagine a teacher who is not bright and well trained in math feeling in control, opportunistic, and unafraid! But the criteria for subject matter should be its mathematical integrity and its relevance to quantitative experiences and questions that are natural to human beings. This leaves a lot of latitude in choosing subject matter. Basic quantitative insights and techniques, well exercised in any of a number of settings, are readily transferable by students, whereas the most enlightened and carefully drawn set of mathematical facts, in the hands of those with little insight, is a dangerous weapon! So math teachers have to have a feeling for math—there's just no substitute for that, which may say more about those we should pursue to become math teachers than it does about what particular mathematics they should learn.

EXAMPLES OF ELEMENTARY SCHOOL MATHEMATICS

That said, I'll tell you what math I would talk about if I were teaching in elementary school. But I want to insist that another choice made by someone else can be equally valid, with equal mathematical integrity and equal relevance to quantitative experiences and questions that are natural to human beings. Let me list four areas of knowledge that I would like children to understand before they leave grade school:

1. Place value, the base 10 number system
2. Operations with fractions
3. Estimating, approximation, margin of error
4. Lengths, areas, and volumes

I'd like to make some comments on each of these areas and to give some examples "at the high end" of what I would like teachers and students to know about them.

Place Value

"10101 = 21." If that equation makes sense to you, you know more about place value than I want children to know and about as much as I'd like most teachers to know. The left side of the above equation is, let's say, Mayan, and the right side is the equivalent phrase in English. The intellectual (mathematical) content of the two phrases is the same—it's just that, by historical accident, people counted with their arms in the land where the Mayan language developed, but they counted with their fingers in the land where English developed (see Figure 4.1).

Any difference between the two columns in Figure 4.1 has about as much to do with mathematics as the differences between good translations of *Brothers Karamazov* into, say, Spanish and French. I'd like teachers to understand that our decimal system is one of several languages with which we can express mathematical ideas, just as English is one of several languages with which we can express poetic ideas. Mathematical notation *does* affect mathematical thought, facilitating some concepts and obscuring others—if you doubt that, just try to do a large multiplication problem using Roman numerals. But there is a fundamental difference between the concepts themselves and the system we use to express them.

Operations with Fractions

The problem "$6 \div 2 = ?$" means "How many 2's are there in 6?" So too, "$\frac{1}{2} \div \frac{2}{9} = ?$" means "How many $\frac{2}{9}$'s are there in $\frac{1}{2}$?" The latter is a problem about halves and ninths. Figure 4.2 shows a suitable model of the number 1 in which we can easily see the halves and ninths of it.

Figure 4.2 produces a graphical approach to solving $\frac{1}{2} \div \frac{2}{9}$. Using "invert and multiply" yields the same answer, but without a concrete referent: $\frac{1}{2} \div \frac{2}{9} = \frac{1}{2} \times \frac{9}{2} = \frac{9}{4}$, or $2\frac{1}{4}$. I would like a fifth-grade math teacher to be

How to add:
When you get 2 in a column replace it with 1 in the column to the left.

Multiplication table:

	0	1
0	0	0
1	0	1

Sample problems

```
  10101
 ×   11
  10101
 10101
 111111
```

```
       1010.1
10 | 10101.0
     10
      010
       10
        010
         10
```

How to add:
When you get 10 in a column replace it with 1 in the next column to the left.

Multiplication table:

(Remember third and fourth grade)

Sample problems

```
  21
 × 3
  63
```

```
    10.5
2 | 21.0
    2
    010
     10
```

FIGURE 4.1: The left-hand column is in Mayan, the right-hand column is the exact translation of the same mathematics into English.

able to make the entire journey, without skipping any of the necessary transitions, from the lesson in Figure 4.2 to the rule of "invert and multiply."

Perhaps calling for the depth of knowledge and understanding necessary to make such a journey implicitly advocates the introduction of "math specialists" to teach fifth- and sixth-grade math. At least there should be one teacher in any given school who can make the journey.

Why do I think that understanding operations with fractions is so important? I guess it's because of my experience teaching mathematics at the college level.

Calculus becomes a memorization game instead of a learning experience because students don't understand algebra. Algebra becomes a memorization game instead of a learning experience because students don't understand fractions. For example,

$$\frac{1}{x} + \frac{1}{y} = \frac{(y+x)}{xy}$$

Is this equality correct? If so, why? If not, why not? This is just adding fractions by finding a common denominator, or, if you want, just multiplying

$$\frac{1}{x} + \frac{1}{y}$$

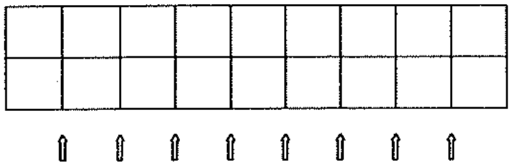
by 1 in a fancy way

$$1 = \frac{xy}{xy}$$

so that

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 1 \times \left(\frac{1}{x} + \frac{1}{y}\right) \\ &= \left(\frac{xy}{xy}\right) \cdot \left(\frac{1}{x} + \frac{1}{y}\right) \\ &= \left(\frac{xy}{xy}\right) \times \left(\frac{1}{x}\right) + \left(\frac{xy}{xy}\right) \cdot \left(\frac{1}{y}\right) \\ &= \left(\frac{y}{xy}\right) + \left(\frac{x}{xy}\right) \\ &= \frac{y+x}{xy}. \end{aligned}$$

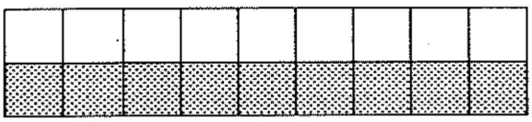
Squint a little bit, see 3 instead of x and 5 instead of y, and you have a problem in adding fractions. If you know how to add fractions with unlike denominators, you know how to do this algebra problem. There is no other area of elementary school mathematics more intimately related with what comes later.



← This line divides 1 into halves

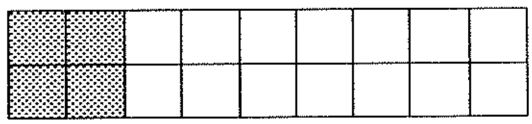
These lines divide 1 into ninths

Here's $\frac{1}{2}$ (of our 1):



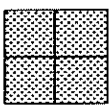
$\frac{1}{2}$

Here's $\frac{2}{9}$ (of that same 1):

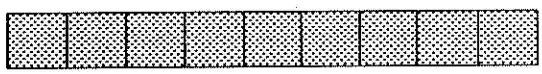


$\frac{2}{9}$

So how many of

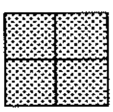


fit into



We'll have to do some cutting and rearranging, but the answer should be

Two and
one-fourth of
these things:



fit into this thing:

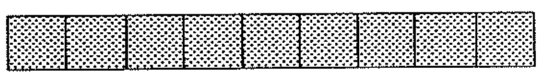


FIGURE 4.2 How many $\frac{2}{9}$'s are in $\frac{1}{2}$?

Estimating

Lest this turn into a math lesson that doesn't know when to stop, I'll combine items 3 and 4 on my list and integrate my example of the type of estimation I find useful with the type of geometry I find useful (see Figure 4.3).

Children should be accustomed enough to estimating size to look at Figure 4.3 and decide that it will take less than 4 cans to paint the inside of the circle, and to give convincing reasons why it will take more than 2 cans. But suppose I wanted to know the answer to within one decimal place. What does that mean? What sort of strategy might I use to get the answer within the desired margin of error? These are fundamental questions about human quantitative experience. The questions are deep, interesting, yet accessible. Learning to deal with them successfully is easily transferable to other quantitative situations. Maybe they also give us a chance to meaningfully touch some of the concepts of higher math in a way accessible to children.

Figure 4.4 shows some useful strategies for the particular problem introduced in Figure 4.3.

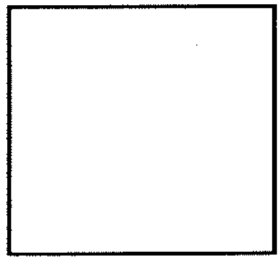
I've tried to pick just a few examples of the kind of mathematical skills I think a math teacher might aim for. Even in these examples, there are many ways to deal with them successfully. There are many ways to teach about them, and many ways to think about them, that have mathematical value and integrity and that prepare children for quantitative success. Any way that gets children to think is good—there are no end of good ways. Good teachers, when they aren't too tired or overburdened, will find the way that is most natural to them.

Do my choices of examples mean that I don't think rote arithmetical skills are important? No, I don't think that at all. It should go almost without saying that kids have to be able to do the traditional computations such as adding, subtracting, multiplying, and dividing whole and decimal numbers with several digits, and do these computations rapidly and accurately. That's the base; those are the calisthenics; you have to do the calisthenics and keep doing them to stay in shape, or you can't play the game. But you also have to play the game, not just do calisthenics!

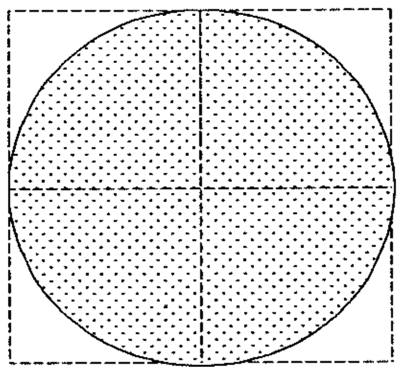
SOME FUNDAMENTAL QUESTIONS

I've tried to give a feel, by example, for the kind of mathematical acumen an elementary school math teacher should have and the kind of mathematical goals he or she might well have for students. It doesn't make sense for me to try to outline an entire curriculum—the National Council of Teachers of Mathematics (1989) has recently done that far better than any of us could. And besides, as I tried to stress at the outset, it's not always the particular topic or

If it takes one can of paint to paint the inside of this square



estimate how many cans of paint it takes to pain the inside of this circle:



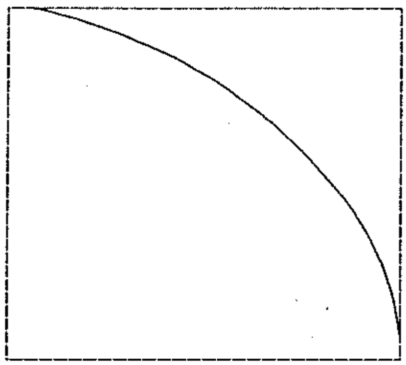
In other words, estimate the value of the number π.

FIGURE 4.3: An estimation problem.

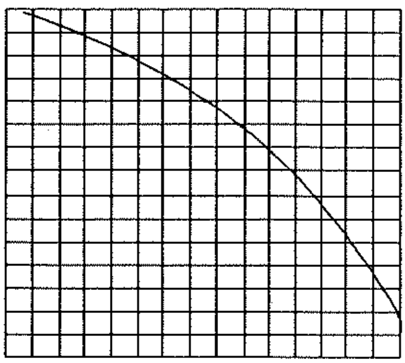
approach that matters most. For me the more fundamental questions are as follows:

Does the subject matter have some mathematical integrity?
Does it have beauty and order?
Does it respond to a quantitative issue that is natural and basic for us humans?
Can this teacher teach it with conviction, and with some feeling for its essence?

1. It is sufficient to solve "one-fourth" of the problem:



2. A fine grid will help:



3. If it takes one can of paint to paint the big square, how much paint does it take to paint one of the tiny squares? (There are 19 in each direction.)
4. How many tiny squares lie entirely inside the quarter-circle?
5. Use this information to give a lower estimate for the amount of paint needed to paint the quarter-circle.
6. How many tiny squares completely cover up the quarter-circle?
7. Use this information to give an upper estimate for the amount of paint needed to paint the quarter-circle.
8. How far is your lower estimate from your upper one? So how close are you to the exact value of one-fourth π?
9. Now multiply your estimates by four. What happens to your margin of error?

FIGURE 4.4: Finding a more precise estimate.

A PROFESSION IN DANGER

Finally, why in the world does a research mathematician, with a safe and comfortable job at a nice university, worry about grade school math? I suppose there are some selfless, noble motives that one could cite, but for many of us the issue is more crass and mundane than that. Our profession is currently in danger, and the danger is directly traceable to the fact that our culture no longer values what we do. Our schools reflect our culture and they transmit its message too—go to school and get training so you can get a job and make money, preferably a lot of money. Worry about training, but don't worry about education, that's too impractical for all but the ivory tower types. Get grades, not ideas.

So we have to import our scientists from other countries, and even that is becoming more difficult as career opportunities for those people increase in their home countries. Our university students complain that their math teachers don't speak English. Of course not, when the students' own older brothers and sisters, and fathers and mothers, can't compete for university teaching positions because of inferior scientific qualifications!

All aspects of mathematics, from grade school to advanced research, develop together, or in the end none develop. Some of us find mathematics exquisitely beautiful, the quantitative equivalent of the best poetry and literature. And if history is any guide, future generations will often find mathematical theory, developed now for aesthetic reasons, astoundingly useful. But the entire enterprise is threatened from within because it is not valued by our young, and we mathematicians share the blame for that. We are often intellectual "yuppies," concentrating only on that which is at the pinnacle, because that's where the personal rewards are, but forgetting to attend to the base. Together, let us attend to the base lest the entire structure crumble!

NOTES

1. Mathematicians understand the square root of -1 to be written as i. So the square root of -2 is $i\sqrt{2}$ [because $(i\sqrt{2})^2 = i^2(2)$, or 2(-1), or -2]. The second statement has to do with expressing the same quantity in different number systems. In this case, twenty-one is represented in base 2 as 10101, where the places are worth, from right to left, respectively: $2^0, 2^1, 2^2, 2^3, 2^4$. Thus 10101 equals $(1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$, or $16 + 0 + 4 + 0 + 1$. In base 10, the places are worth, from right to left, respectively, 10^0, 10^1, 10^2, 10^3, and so on, so twenty-one is represented as 21 because it's $(2 \times 10^1) + (1 \times 10^0)$, or $20 + 1$.

REFERENCES

National Council of Teachers of Mathematics, Commission on Standards for School Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.